On the Equivalence of Relations $K_{q_1q_2m}$.

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Introduction.

T. H. Hildebrandt* has developed the theory of systems $(\mathfrak{Q}; K)$ where K is a relation $K_{q_1q_2m}$.† In terms of a relation K a relation L (limit) is defined. This leads, through the conditioning of the K relation, to a basis system $(\mathfrak{Q}; K^{1367})$ which is found to be an adequate substitute for the system $(\mathfrak{Q}; V)$ of Fréchet, where V denotes a voisinage.‡ In a system $(\mathfrak{Q}; K^{1367})$§ the relation K may be unsymmetric in the arguments q_1, q_2; and the identical sequence consisting of a single repeated element q may not have q as a limit. This constitutes a reduction in the hypothesis of Fréchet.

It is shown in the present paper (§ 6) that from any system $(\mathfrak{Q}; K)$ a system $(\mathfrak{Q}; \dot{\bar{K}})$ may be defined such that $\dot{\bar{K}}$ is symmetric and the identical sequence for q always has q for a limit. If K^{167}, every continuous function in the system $(\mathfrak{Q}; K)$ is continuous in the derived system $(\mathfrak{Q}; \dot{\bar{K}})$. The systems $(\mathfrak{Q}; K)$, $(\mathfrak{Q}; \dot{\bar{K}})$ are said to be equivalent with respect to continuity.

The form of equivalence holding between K and $\dot{\bar{K}}$ follows from the relation between L and $\dot{\bar{L}}$ derived from K and $\dot{\bar{K}}$, respectively. The general theory of equivalences of systems determined by limit relations is important, and is developed, together with some of its consequences, in §§ 1, 2. Two systems $\tilde{\Sigma}$, $\hat{\Sigma}$ containing a class $\mathfrak{Q}$ and defining limit in $\mathfrak{Q}$ may have one of three types of equivalence, denoted by (A), (B), (C). For example $(\mathfrak{Q}; K^{167})$, $(\mathfrak{Q}; \dot{\bar{K}}^{167})$ are equivalent (B). Each of the three types of equivalence implies its predecessor.

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*"A Contribution to the Foundations of Fréchet's Calcul Fonctionnel," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 237–290.

† This is a relation K_2, introduced by E. H. Moore, "Introduction to a Form of General Analysis." *The New Haven Mathematical Colloquium*, New Haven (1910), p. 126.

‡ *Rendiconti del Circulo di Palermo*, Vol. XXII, pp. 1–64.

§ Eight properties of a K relation are defined by Hildebrandt, *loc. cit.*, pp. 243–244. The presence of property i is denoted by K^i. The notation and results of Hildebrandt's memoirs are presupposed in the present paper.

In § 4 necessary and sufficient conditions for equivalence (B) or (C) of systems $(\mathfrak{Q}; \tilde{K}^1)$, $(\mathfrak{Q}; \hat{K}^1)$ are obtained. These conditions are applied in § 5 to show that a relation K^{167} (unsymmetric) is equivalent (C) to a relation $\bar{K}^{125}$ (symmetric), and also in § 6 to show that a relation K^{1367} is equivalent (B) to a relation $\dot{\bar{K}}^{12345}$, therefore equivalent (B) to a voisinage. Extensions of theorems of Fréchet obtained by Hildebrandt, which involve the hypotheses K^{1367}, are immediate consequences of this theory of equivalence.*

1. Denote by $S \equiv \{q_n\}$ an infinite sequence of elements of a class $\mathfrak{Q}$. We shall understand by the notation qLS that q is a limit of S. If S consists of a single repeated element q, S is the identical sequence Iq for q. We consider systems Σ containing a class $\mathfrak{Q}$ relative to which, for every element q and sequence S, it is determined, whether or not qLS, L being defined in some manner in the system Σ.

(A) *Two systems $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent with respect to limiting element if the derived class $\mathfrak{R}'$ of any subclass $\mathfrak{R}$ of $\mathfrak{Q}$ is the same in $\tilde{\Sigma}$ as in $\hat{\Sigma}$.*

(B) *$\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent with respect to continuity if for every sequence S and element q, such that Iq is not a subsequence of S, q is a limit of S in either system if it is a limit of S in the other.*

(C) *$\tilde{\Sigma}$; $\hat{\Sigma}$ are equivalent with respect to limit of a sequence if every limit in one system is a limit in the other.*

The significance of the notation; $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (B) [equivalent (C)], is evident. Equivalence (B) implies equivalence (A) and is in turn implied by equivalence (C). While it is desired to call attention to the large body of theorems which equivalence (A) will carry over from a system $\tilde{\Sigma}$ to a system $\hat{\Sigma}$, nevertheless the principal results of this paper relate to equivalences (B) and (C).

If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (B) and the respective limit relations $\tilde{L}$, $\hat{L}$ are of the type L^2 † then $q\tilde{L}S$ implies $q\hat{L}S'$, if S' is any subsequence of S which does not contain Iq. A similar statement holds for $q\hat{L}S$.

Two sequences S', S'' are complementary subsequences of S if every element of S not in S' is in S''. A relation L is a relation L^7 if qLS', qLS'' imply qLS.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent with respect to continuity and $\tilde{L}^{2467}$, $\hat{L}^{2467}$; then $\tilde{L}$, $\hat{L}$ are identical.

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* For example, the theorems of Hildebrandt, *loc. cit.*, § 22, pp. 288–290.

† A relation L is of type L^2 in case qLS implies qLS', where S' is any subsequence of S. Properties 1–6 of a relation L are defined by Hildebrandt, *loc. cit.*, pp. 281–282.

We must show that, given $q\tilde{L}S$, $q\hat{L}S$ follows. If S does not contain Iq, $q\hat{L}S$ is implied by the definition of equivalence with respect to continuity. If, except for a finite set of elements, S coincides with Iq, we have from $\tilde{L}^2$, $q\tilde{L}Iq$, and from $\hat{L}^6$, $q\hat{L}Iq$. Hence from $\hat{L}^4$, $q\hat{L}S$. Otherwise S contains Iq and an infinite complementary sequence S' which does not contain Iq. Therefore $q\hat{L}Iq$, $q\hat{L}S'$; and it follows from $\hat{L}^7$ that $q\hat{L}S$. A repetition of the argument with $\tilde{L}$, $\hat{L}$ interchanged completes the proof.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity and* $\tilde{L}^2$, $\hat{L}^2$, *every function* μ continuous on $\mathfrak{Q}$ in either system is continuous in the other.*

We have to show, for example, that if μ is continuous on $\mathfrak{Q}$ in $\tilde{\Sigma}$ and $q\hat{L}S$, $(S\equiv\{q_n\})$ then $L_n\mu_{q_n}=\mu_q$. If S differs from Iq only in a finite number of elements, the result is immediate. Otherwise S will contain S' complementary to Iq. From $\hat{L}^2$, $q\hat{L}S'$; and from equivalence $q\tilde{L}S'$. Therefore $L_n\mu_{q'_n}=\mu_q$, and we have immediately $L_n\mu_{q_n}=\mu_q$.

2. The following properties of classes are definable in terms of limiting element: compact, closed, interior, perfect, separable. Hence,

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (A) [*with respect to limiting element*] *the properties compact, closed, etc., of classes have the same significance in one system as in the other.*

That is, for example, if $\mathfrak{R}$ is compact in $\tilde{\Sigma}$ it is compact in $\hat{\Sigma}$.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (B) [*with respect to continuity*] *and* $\tilde{L}^2$, $\hat{L}^2$ *then in addition to the preceding proposition every function continuous in one system is continuous in the other.*

If $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent* (C) [*with respect to limit of a sequence*] *and* $\tilde{L}^2$, $\hat{L}^2$ *we have the results above and the further result that* $\tilde{L}$ *and* $\hat{L}$ *are identical relations.*

3. In terms of a relation K we define a relation L (limit) as follows:† given a sequence $\{q_n\}$ and an element q such that for every integer m there exists an integer n_m such that $n = m_m$ implies

$$K_{q_n q m},$$

q is said to be a limit of the sequence $\{q_n\}$, and we write

$$q = \underset{n}{L} q_n .$$

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* The relation (B) between $\tilde{L}$ and $\hat{L}$ derives its name from the fact expressed in this proposition.

† Cf. Hildebrandt, *loc. cit.*, p. 249. This definition differs from that of Fréchet in terms of voisinage only in non-essentials.

The relation L so defined is a relation L^K (dependent on K), and is a relation L^{23457}.* Unless K is conditioned *limit* is not necessarily unique. It is evident that from different relations K, distinct relations L may be defined for a single class $\mathfrak{Q}$.

It will be noticed that in case the relation K is unsymmetric, the relations K_{q_nqm} and K_{qq_nm} may have different significance, and that if we replace K_{q_nqm} by K_{qq_nm} in the definition above a different relation L may result. A number of relations L may be defined for K unsymmetric which reduce to the same relation under the hypothesis of symmetry. The following example will serve to make clear the significance of these remarks. Let $\mathfrak{Q}$ denote the interval $0 \leqq q \leqq 1$. The relation $K_{q_1q_2m}$ holds if $q_1 \leqq q_2$ and $|q_1 - q_2| = \frac{1}{2^m}$. Then $q = \underset{n}{L} q_n$ implies that the elements q_n all lie to the left of q, if n is sufficiently large; *i. e.*, q is a right-hand limit of the sequence $\{q_n\}$. The relation K_{qq_nm} may hold for every m and $n \geqq n_m$, but q will not be a limit of $\{q_n\}$ in this sense.

If we denote by $\underline{K}$ a relation such that

$$K_{q_1q_2m} \textit{ implies } \underline{K}_{q_2q_1m},$$

we call $\underline{K}$ the *conjugate* of K. The limit relation $\underline{L}$ will be the conjugate of the relation L defined in terms of K. In the example above the relations L, $\underline{L}$ are distinct, since $\underline{K}_{q_1q_2m}$ implies $q_1 \leqq q_2$, and therefore $q = \underset{n}{\underline{L}} q_n$ implies q is a left-hand limit.

4. Denote by $\phi(q, m)$ any integral-valued single-valued function of two arguments q, m such that for every q

$$\underset{m}{L} \phi(q, m) = \infty.$$

In terms of these functions ϕ we state the theorem:

THEOREM I. *A necessary and sufficient condition that two systems* $(\mathfrak{Q}; \hat{K}^1)$,† $(\mathfrak{Q}; \tilde{K}^1)$ *be equivalent with respect to limit of a sequence is that there exist a function* ϕ, *and for every element* q_2 *an* m_2, *such that for every* $m \geqq m_2$,

$$\hat{K}_{q_1q_2m} \textit{ implies } \tilde{K}_{q_1q_2\phi(q_2, m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \textit{ implies } \hat{K}_{q_1q_2\phi(q_2, m)}; \tag{b}$$

furthermore, if in each case q_1 *is supposed to be distinct from* q_2, *conditions* (a) *and* (b) *become necessary and sufficient for equivalence of* $(\mathfrak{Q}; \hat{K}^1)$ $(\mathfrak{Q}; \tilde{K}^1)$ *with respect to continuity.*

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* Cf. Hildebrandt, *loc. cit.*, p. 249. That L^7 is an immediate consequence of the definition of limit.

† A relation K is a relation K^1 if $K_{q_1q_2m}$ implies $K_{q_1q_2m'}$, where m' is any integer less than m.

A proof of the first part of the theorem follows. A slight modification of this proof serves to establish the second part of the theorem.

We will establish the sufficiency of the condition by showing that $\hat{L}$ and $\tilde{L}$ are identical relations. From the symmetrical nature of the situation it will be sufficient to show that if $q = \underset{n}{\hat{L}} q_n$ then $q = \underset{n}{\tilde{L}} q_n$.

By definition, from $q = \underset{n}{\hat{L}} q_n$ we have for every integer m_0 an integer n_{m_0} such that $n \geqq n_{m_0}$ implies

$$\hat{K}_{q_n q m_0}.$$

Condition (a) gives for $m_0 \geqq m_2$ (dependent on q)

$$\tilde{K}_{\bar{q}_n q \phi(q, m_0)}.$$

Since $\underset{m}{L} \phi(q, m) = \infty$ we have for every m an m_0 such that $\phi(q, m_0) \geqq m$. If we choose $n_m \geqq (n_m, n_{m_2})$ we have for every $n \geqq n_m$, because of $\tilde{K}^1$ and the preceding relation,

$$\tilde{K}_{q_n q m}.$$

Therefore $q = \underset{m}{\tilde{L}} q_n$, as was to be proved.

We complete the proof by showing that conditions (a) and (b) are necessary consequences of the definition of equivalence for systems $(\mathfrak{Q}; K)$. Denote by $\hat{\mathfrak{Q}}_{q_2 m}$ the class of all elements q_1 in the relation $\hat{K}_{q_1 q_2 m}$ but not in the relation $\hat{K}_{q_1 q_2 (m+1)}$. Denote by $\tilde{m}_{q_1}$ the greatest value of m for which the relation $\tilde{K}_{q_1 q_2 m}$ holds, in case such greatest value exists.

We now define $\tilde{\phi}(q_2, m)$ as follows: If among the values $\tilde{m}_{q_1}$ for elements q_1 of $\hat{\mathfrak{Q}}_{q_2 m}$ there is a least, take $\tilde{\phi}(q_2, m)$ equal to this $\tilde{m}_{q_1}$. If there is no least $\tilde{m}_{q_1}$ there is either no $\tilde{m}_{q_1}$ or there is a q_1 such that $\tilde{m}_{q_1} < 1$. If there is no $\tilde{m}_{q_1}$, take $\tilde{\phi}(q_2, m) = m$. In the remaining case $\tilde{\phi}(q_2, m) = \tilde{m}_{q_1} < 1$. It is desired to prove that for every q_2, $\underset{m}{L} \tilde{\phi}(q_2, m) = \infty$. Suppose there exists $\overline{m}$ such that for every m, $\phi(q_2, m) < m$. Consider first the values of m for which $\tilde{\phi}(q_2, m) = \tilde{m}_{q_{1m}}$ for some q_{1m} in $\hat{\mathfrak{Q}}_{q_2 m}$. From $\hat{K}^1$ it follows that either every m is of this type, or else there is a least m for which there exists a q_{1m} in $\hat{\mathfrak{Q}}_{q_2 m}$ which possesses an $\tilde{m}_{q_1}$. In the first case $q_2 = \underset{m}{L} q_{1m}$. From the equivalence assumed in the hypothesis, $q_2 = \underset{m}{L} q_{1m}$. This contradicts the assumption that all the $m_{q_{1m}}$ are less than $\overline{m}$. In the second case there exists an m_0 such that for every $m > m_0$, $\tilde{\phi}(q_2, m) = m$, and therefore $\underset{m}{L} \phi(q_2, m) = \infty$. The assumption $\tilde{\phi}(q_2, m) < \overline{m}$ is therefore contradicted in every case.

We define a number $\tilde{m}_2$ as follows: Either $\tilde{\phi}(q_2, m) = \tilde{m}_{q_1m}$ for every m, or there exists m_0 such that for $m' \geqq m_0$, $\tilde{\phi}(q_2, m') = m'$. In the first case take $\tilde{m}_2$ such that $\tilde{\phi}(q_2, \tilde{m}) \geqq 1$; in the second case take $\tilde{m}_2 = m_0$. Then it is only necessary to take account of $\tilde{K}^1$, and the definitions of $\tilde{m}_2$, $\tilde{m}_{q_1}$, $\tilde{\phi}(q_2, m)$ to see that for every $m \geqq m_2$ condition (a) is satisfied.

By means of a similar choice of $\hat{m}_2$, $\hat{\phi}(q_2, m)$ we may satisfy condition (b). Choosing $m_2 \geqq (\tilde{m}_2, \hat{m}_2)$ and $\phi(q_2, m)$ equal to the lesser of $\tilde{\phi}(q_2, m)$, $\hat{\phi}(q_2, m)$ we have m_2 and ϕ (dependent on q) as required. This completes the proof of the first part of Theorem I.

The hypothesis of equivalence with respect to limit of sequence of K^1 and $\underline{K}^1$ (conjugate of K) therefore leads to the existence of the function ϕ of Theorem I. We show in § 5 that if K^1 and $\underline{K}^1$ are equivalent (C) they are equivalent (C) to a symmetric relation $\overline{K}^1$ derived from K^1. In view of the fact that L and $\underline{L}$ may be distinct for unsymmetric relations K, while their identity implies equivalence (C) to a symmetric K it would seem desirable to confine attention to the symmetric relations in applications of K relations to the study of classes and functions. Especially when the resulting simplification of the treatment is considered.

There may exist a function ϕ, effective in Theorem I whose values are independent of q_2. We thus obtain a corollary to this theorem:

COROLLARY: *A sufficient condition that two systems* $(\mathfrak{Q}; \hat{K}^1)$ $(\mathfrak{Q}; \tilde{K}^1)$ *be equivalent* (C) *is that there exists a function* ϕ, *such that for every* m *and* q_1, q_2

$$\hat{K}_{q_1q_2m} \text{ implies } \tilde{K}_{q_1q_2\phi(m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \text{ implies } \hat{K}_{q_1q_2\phi(m)}. \tag{b}$$

That these conditions are not necessary will be seen if we suppose $\mathfrak{Q}$ represents the open interval $0 \leqq q \leqq 1$ and define $\hat{K}, \tilde{K}$ as follows:

For $m < 1$, every pair of elements q_1q_2 is in the relations $\hat{K}_{q_1q_2m}$, $\tilde{K}_{q_1q_2m}$. For $m \geqq 1$, we take

$$\hat{K}_{q_1q_2m} \equiv |q_1 - q_2| \leqq \frac{1}{2^m}.^*$$

If $(k-1)/k = k/(k+1)$ then

$$\tilde{K}_{q_1q_2m} \equiv |q_1 - q_2| \leqq \frac{1}{(k+1)^m}.$$

From the definitions $\hat{K}, \tilde{K}$ are symmetric. They are easily seen to be equivalent (C), but the hypothesis of existence of a function ϕ, leads to a contradiction.

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* That is, the relation $K_{q_1q_2m}$ is defined by this inequality for every pair of elements q_1, q_2.

5. Given an unsymmetric relation K we may define a symmetric relation $\overline{K}$ as follows:

A pair of elements q_1, q_2 are in the relation $\overline{K}_{q_1q_2m}$ if and only if they are in one of the relations $K_{q_1q_2m}, K_{q_2q_1m}$.

It is easy to see that if K^i $(i=1, 2, 3, 4)$ then $\overline{K}^i$ also.

Theorem II. K^{167} *implies* $\overline{K}^{125(678)}$.*

The properties (6), (7), (8) are each equivalent to 5 because of $\overline{K}^2$. We have to show that there exists a function ϕ such that

$$\overline{K}_{q_1q_2m},\ \overline{K}_{q_2q_3m}\ \textit{imply}\ \overline{K}_{q_1q_3\phi(m)}.$$

Now $\overline{K}_{q_1q_2m}, \overline{K}_{q_2q_3m}$, by definition of $\overline{K}$, imply that one of the following four pairs of relations is holding:

$$\begin{aligned} &K_{q_1q_2m}, \quad K_{q_2q_3m};\\ &K_{q_1q_2m}, \quad K_{q_3q_2m};\\ &K_{q_2q_1m}, \quad K_{q_2q_3m};\\ &K_{q_2q_1m}, \quad K_{q_3q_2m}.\end{aligned}$$

If we choose a function ϕ effective in each of the four instances 5, 6, 7, 8, as is possible when K^1, and recall that K^{167} implies K^{58},† we obtain

$$\overline{K}_{q_1q_3\phi(m)}.$$

This, because of the definition of $\overline{K}$, implies

$$\overline{K}_{q_1q_3\phi(m)}.$$

The following proposition is now evident:

K^{167} *implies that there exists a function* ϕ *such that* $\overline{K}_{q_1q_2m}$ *implies* $K_{q_1q_2\phi(m)}$ *and* $K_{q_2q_1\phi(m)}$.

We have shown that from an unsymmetric relation K^1 a symmetric relation $\overline{K}^1$ may always be defined. The converse is not true. Suppose that for every m and q_1, q_2 the relation $\overline{K}_{q_1q_2m}$ holds. $\overline{K}$ is symmetric. It is evident that a relation K^1 can be made unsymmetric only by adding new relations $K_{q_1q_2m}$ to those already established. But this is impossible in the case cited.

Assuming that for at least one pair $\bar{q}_1, \bar{q}_2$ there is an m_0 such that for $m \geqq m_0$ the relation $\overline{K}_{\bar{q}_1\bar{q}_2m_1}$ does not hold, we define K unsymmetric by assuming that the relation $\overline{K}_{q_1q_2m}$ implies $K_{q_1q_2m}$ and in addition $K_{\bar{q}_1\bar{q}_2(m_0+1)}$, but not $K_{\bar{q}_2\bar{q}_1(m+1)}$. Then $\overline{K}^i$ implies K^i where i denotes any of the properties (1), (2), (3), (4), (5), (6), (7), (8).

* For definitions of the properties 1–8 of a K relation see Hildebrandt, *loc. cit.*, pp. 244–246.

† Hildebrandt, *loc. cit.*, p. 246.

In § 4 it was remarked that if K^1 and $\underline{K}^i$ (conjugate) were equivalent (C) they would be together equivalent (C) to a symmetric relation K. This relation is in fact $\bar{K}$. Since K, $\underline{K}$ are equivalent (C) we have (Theorem I) for every q_2 an m_2 and a function ϕ, such that for every $m \geqq m_2$

$$K_{q_1 q_2 m} \textit{ implies } \underline{K}_{q_1 q_2 \phi(q_2, m)} \textit{ which implies } K_{q_2 q_1 \phi(q_2, m)}, \tag{a}$$

$$\underline{K}_{q_1 q_2 m} \textit{ implies } K_{q_1 q_2 \phi(q_2, m)} \textit{ which implies } \underline{K}_{q_2 q_1 \phi(q_2, m)}. \tag{b}$$

To demonstrate the equivalence (C) of K and $\bar{K}$, we choose m_2 so that (a) and (b) are satisfied, and $\bar{\phi}(q_2, m) = (\text{lesser of } m, \phi(q_2, m))$. Then the relations:

$$K_{q_1 q_2 m} \textit{ implies } \bar{K}_{q_1 q_2 \bar{\phi}(q_2, m)},$$
$$\bar{K}_{q_1 q_2 m} \textit{ implies } K_{q_1 q_2 \bar{\phi}(q_2, m)},$$

are easily seen to hold for all $m \geqq m_2$. The equivalence (C) of $\bar{K}$ and $\underline{K}$ is similarly established.

THEOREM III. *Any system* $(\mathfrak{Q}; K^{167})$ *is equivalent to a system* $(\mathfrak{Q}; K^{125})$.

This theorem is a consequence of Theorems I and II. To demonstrate the application of Theorem I we must show that conditions (a), (b) are fulfilled. It is sufficient to remark that $\bar{\phi}_m \leqq (m, \phi_m)$, where ϕ is effective in the proposition of § 5, and $m_2 = 1$, are effective in this instance.

In view of this theorem and § 2, it follows that the theorems of Hildebrandt,* which involve the hypothesis K^{167} are not more general than the corresponding theorems with hypothesis K^{125}, that is the hypothesis of an unsymmetrical K does not in this instance lead to more general results.

6. From K we obtain $\dot{K}^4$ as follows:

$$K_{q_1 q_2 m} \textit{ implies } \dot{K}_{q_1 q_2 m}(m); \tag{a}$$

$$\textit{for every } q \textit{ and } m \textit{ we have } \dot{K}_{qqm}. \tag{b}$$

Then $\dot{K}_{q_1 q_2 m}$ implies either $K_{q_1 q_2 m}$ or $q_1 = q_2$. It is easy to see that K^i implies $\dot{K}^i$ where i is any of the properties (1), (2), ..., (8). Furthermore, K^1 and $\dot{K}^1$ are equivalent (B). In fact $\phi(q_1', m) = m$ is effective in conditions (a), (b) of Theorem I.

THEOREM IV. *Any system* $(\mathfrak{Q}; K^{1367})$ *is equivalent* (B) *to a system* $(\mathfrak{Q}; K^{12345})$.

From Theorem III, K^{167} is equivalent (C) to $\bar{K}^{125}$. But $\bar{K}^{1235}$ is equivalent (B) to $\dot{\bar{K}}^{12345}$. Hence K^{1367} is equivalent (B) to $\dot{\bar{K}}^{12345}$.

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* *Loc. cit.*, pp. 273–290.

Hildebrandt has shown* that the concepts K^{12345} and voisinage (V) are equivalent. Therefore, we have:

THEOREM V. *Any system* $(\mathfrak{Q}; K^{1367})$ *is equivalent* (B) *to a system* $(\mathfrak{Q}; V)$ *of Fréchet, and furthermore, any system* $(\mathfrak{Q}; K^{13467})$ *is equivalent* (C) *to a system* $(\mathfrak{Q}; V)$.

As an application of this theorem we call attention to the theorem: †

In a system $(\mathfrak{Q}; K^{1367})$ *a necessary and sufficient condition that every continuous function on* $\mathfrak{R}$ *be bounded on* $\mathfrak{R}$ *is that* $\mathfrak{R}$ *be extremal,*

which may be obtained from Hahn's ‡ extension of a theorem of Fréchet in the following manner. The condition is necessary in a system $(\mathfrak{Q}; V)$. A system $(\mathfrak{Q}; K^{1367})$ is equivalent (B) to a system $(\mathfrak{Q}; V)$. Therefore every function continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; V)$ is continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; K)$. If every function continuous on $\mathfrak{R}$ in $(\mathfrak{Q}; K)$ is bounded on $\mathfrak{R}$ then $\mathfrak{R}$ is extremal in $(\mathfrak{Q}; V)$. From § 2 $\mathfrak{R}$ is extremal in $(\mathfrak{Q}; K)$, which was to be proved. The condition is sufficient since $\mathfrak{R}$ extremal in a system $(\mathfrak{Q}; K)$ is extremal in any system equivalent (B) to $(\mathfrak{Q}; K)$, that is in $(\mathfrak{Q}; V)$.

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* *Loc. cit.*, p. 248.

† Cf. Hildebrandt, *loc. cit.*, pp. 288–290.

‡ Cf. *Monatshefte für Math. u. Physik,* Vol. XIX, p. 251 ff.